Tarski's Decision Procedure for FO Theory of Reals

November 27, 2012

A brief history

In the 1930's, A. Tarski proved that a solution formula exists for any input formula by giving an algorithm for its construction [Tarski(1951)].

Seidenberg [Seidenberg(1954)] in 1954 and Cohen [Cohen(1969)] in 1969 both offered alternative methods for solving the problem.

However, in 1973 Collins discovered a totally new method [Collins(1975)] based on cylindrical algebraic decomposition (CAD) that was far more efficient than any previous approach. The method has time complexity doubly exponential in the number of variables in F, but polynomial in the number of polynomials in F, the degree of the polynomials in F, the bit length of coefficients in F, and the number of atomic formulas in F.

The methods of Renegar [Renegar(1992)] and of Heintz et. al. [Heintz et al.(1990)Heintz, Roy, and Solernó] are both doubly exponential in the number of quantifier alternations rather than the number of variables.

Other research has focused on the consideration of restrictions of the full theory of real closed fields. For example, Weispfenning [Weispfenning(1988)] considered formulas in which bound variables appear only linearly. Hong [Hong(1993)] treated the case of an input formula of the form

$$(\exists x)[a_2x^2 + a_1x + a_0 = 0 \quad \wedge \quad F']$$

where F' is a quantifier free formula. Finally, Weispfenning [Weispfenning(1994),Weispfenning(1997)] has developed a method for attacking certain formulas in which the quantified variables appear at most cubically.

On Real Closed Fields

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(2) Any elementary predicate in the theory of real–closed fields is equivalent to a quantifier–free one.

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Related areas of active research : Algorithms for quantifier elimination, tools that implement quantifier elimination .

What are Elementary predicates?

The smallest class containing the atomic predicates and closed under $\neg, \vee, \wedge$ and the quantifiers:
$(\forall x)P(x, y_1, ..., y_n)$ ("$P(x, y_1, ..., y_n)$ holds for all x") and
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The class of quantifier–free elementary predicates is defined as the smallest class containing the atomic predicates which is closed under $\neg, \vee, \wedge$, no quantifiers being used.

What do we mean by "equivalent to"?

Two elementary predicates $P(x_1, ..., x_n)$ and $Q(x_1, ..., x_n)$ are said to be equivalent in the theory of algebraically closed fields if for any algebraically closed field F and elements $a_1, ..., a_n \in F$, when $P(a_1, ..., a_n)$ is true if and only if $Q(a_1, ..., a_n)$ is true.

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P and Q are equivalent in the theory of real– closed fields if for any real–closed field F and elements $a_1, ..., a_n \in F, P(a_1, ..., a_n)$ is true if and only if $Q(a_1, ..., a_n)$ is true.

Corollary 1.1. Let $F \subseteq E$ be algebraically closed fields and let $P(x_1, ..., x_n)$ be an elementary predicate in the theory of algebraically closed fields. If $a_1, ..., a_n$ are elements of F then $P(a_1, ..., a_n)$ is true in F if and only if it is true in E.

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Corollary 1.2 (Elementary Lefschetz Principle). Let S be an elementary statement in the theory of algebraically closed fields. If S is true for one algebraically closed field F then S is true in all algebraically closed fields having the same characteristic as F .

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Corollary 1.3 (Tarski Principle). Let S be an elementary statement in the theory of real–closed fields. If S is true for one real–closed field F then S is true in all real–closed fields.

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Lemma 2.1. Let P be a quantifier–free predicate constructed from atomic predicates $A_1, ..., A_n$. Then P is equivalent to a disjunction $P_1 \vee P_2 \vee ... \vee P_m$ where each P_i has the form $B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{ir_i}$ with each B_{ij} of the form A_k or $\neg A_k$.

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Corollary 2.2.

- A quantifier–free predicate in the theory of fields is equivalent to the disjunction of predicates of the form $f_1 = 0 \wedge \ldots \wedge f_p = 0 \wedge g_1 \neq 0 \wedge \ldots \wedge g_q \neq 0$.

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Corollary 2.3 It will suffice to prove the elimination of quantifiers for predicates of the form $(\exists x)[f_1 = 0 \wedge \ldots \wedge f_p = 0 \wedge g_1 > 0 \wedge \ldots \wedge g_q > 0]$ in the real–closed case and $(\exists x)[f_1 = 0 \wedge \ldots \wedge f_p = 0 \wedge g_1 \neq 0 \wedge \ldots \wedge g_q \neq 0]$ in the algebraically closed case.

Pseudomonic Form

A quantifier–free predicate $P(x)$ is in pseudomonic form (relative to the variable x) if it has the form $c \neq 0 \wedge Q(x)$ where c is a polynomial not involving x and divisible by the leading coefficients (with respect to x) of all polynomials occurring in $Q(x)$

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Corollary 3.2. A quantifier–free predicate in the theory of fields is equivalent to the disjunction of pseudo–monic predicates of the form $c \neq 0 \wedge f_1 = 0 \wedge \ldots \wedge f_p = 0 \wedge g_1 \neq 0 \wedge \ldots \wedge g_q \neq 0$ where c is divisible by the leading coefficients of the f_i and the g_j.

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A quantifier–free predicate in the theory of ordered fields is equivalent to the disjunction of predicates of the form $c \neq 0 \wedge f_1 = 0 \wedge ... \wedge f_p = 0 \wedge g_1 > 0 \wedge ... \wedge g_q > 0$ where c is divisible by the leading coefficients of the f_i and the g_j .

In proving the theorems we will take the following as our induction hypothesis.

Induction Hypothesis(n). Let $P(x)$ be a quantifier–free predicate and let f be a polynomial of degree at most n in x. Let c be a polynomial in the variables other than x which is divisible by the leading coefficient of f . Then $(\exists x)[c \neq 0 \wedge f = 0 \wedge P(x)]$ is equivalent to a quantifier–free predicate.

Lemma 4.1 . If the induction hypothesis holds for n then the induction hypothesis holds for $n+1$ provided that any predicate $(\exists x)[c \neq 0 \wedge f = 0 \wedge g_1 > 0, ...g_q > 0]$ with deg $f = n+1$, deg $g_i \leq n$ for all i, and c divisible by the leading coefficients of f and all g_i , is equivalent to a quantifier–free predicate.

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Lemma 4.2. If the induction hypothesis holds for all n and if any predicate of the form $(\exists x)[c \neq 0 \wedge g_1 > 0, ...g_q > 0]$ with c divisible by the leading coefficients of all g_i is equivalent to a quantifier–free predicate, then any predicate $(\exists x)P(x)$ with $P(x)$ quantifier–free is equivalent to a quantifier–free predicate.

The real–closed case

Lemma 5.1. Suppose the induction hypothesis holds for n. Then a predicate $(\exists x)[c \neq 0 \wedge g_1 > 0, \ldots g_q > 0]$ with deg $g_i \leq n$ for all i, and c divisible by the leading coefficients of all g_i , is equivalent to a quantifier–free predicate.

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Lemma 5.2. Suppose the induction hypothesis holds for n. Then a predicate $(\exists x)[c \neq 0 \wedge f = 0 \wedge g_1 > 0, ... g_q > 0]$ with deg $f = n + 1$, $g_i \leq n$ for all i, and c divisible by the leading coefficients of all g_i , is equivalent to a quantifier–free predicate.

Lemma 5.3 Suppose the induction hypothesis holds for n. Let deg $g \leq n$ and let c be a polynomial in the other variables divisible by the leading coefficient of g. Let y and z be variables not occurring in g. Then the following assertions are equivalent to quantifier–free predicates in y, z, and the remaining variables.

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4. $c \neq 0$ and g is never zero.

Corollary 5.4. Let c be a polynomial in the variables other than x and let $f(x)$ be a polynomial whose leading coefficient divides c. Then the assertions

(1) $c \neq 0$ and $f(x) > 0$ for $x >> 0$

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are equivalent to quantifier–free predicates in the coefficients of f. In fact (1) is equivalent to $c \neq 0 \wedge a_0 > 0$ while (2) is equivalent to $c \neq 0 \wedge (-1)^n a_0 < 0$.

Lemma 5.5. The predicate $(\exists x)[c \neq 0 \wedge g1 > 0 \wedge ... \wedge g_q > 0]$ with c divisible by the leading coefficients all g_i , is equivalent to the disjunction of the following predicates where i, j, and k run from 1 to q.

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That's all folks!